ABSTRACT
The aim of this study is to investigate the motion of a two-axis single-link flexible manipulator. In particular, the motion in the vertical plane is modelled, and a mixed sensitivity H\(\infty\) controller is designed. This controller is found to give a better tracking response than a classical controller, and can be designed to be robust to payload variation. However, careful choice of weighting functions is required to ensure good performance.

INTRODUCTION
A robot manipulator which has been designed to reduce weight and improve speed, or to have a particularly long reach, is liable to exhibit significant structural flexibility in one or more links of the arm. If neglected this will cause vibration and static deflection. Thus the flexibility must be modelled and controlled if the manipulator design is to be viable.

Many researchers have addressed the problem of controlling flexible manipulators over the last twenty years. Initial studies concentrated on the motion of a single link flexible arm driven in the horizontal plane. Latterly multi-link manipulators have been investigated, still largely in the horizontal plane; this presents a complex non-linear vibration control problem.

In the present study a two-axis single link manipulator is considered, driven about a revolute hub in both vertical and horizontal planes. The objectives are:

- to model the manipulator, including any interactions between horizontal and vertical motions
- to design a robust H-infinity controller
- to tackle the particular difficulties encountered in controlling the manipulator in the vertical plane under the influence of gravity.

This paper focusses on motion in the vertical plane, and follows on from simulation results contained in [1].

A number of H\(\infty\) controller design approaches have been proposed for flexible manipulators. Lenz et al [2] developed a mixed sensitivity H\(\infty\) controller that showed considerable promise. The majority of the work that has been conducted has concentrated on single payload loading conditions. However, Cashmore [3] used an uncertainty model to take into account some variation in payload conditions so utilising the robustness of the H\(\infty\) controller. Recent work on H\(\infty\) controllers for flexible manipulators has been published by Matsuno & Tanaka [4], Estiko et al [5], Landau et al [6], and Jovik & Lennartson [7]. All of these authors highlight the difficulties associated with choosing the weighting functions.

EXPERIMENTAL FLEXIBLE MANIPULATOR
An experimental single link two degree of freedom flexible manipulator has been constructed (Figures 1 and 2). The two degrees of freedom allow the end effector to be moved in the horizontal and vertical planes. Table 1 contains component specifications.
TABLE 1: Experimental manipulator specifications

<table>
<thead>
<tr>
<th>Flexible link</th>
<th>Length (L)</th>
<th>1.00 m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mass (M&lt;sub&gt;b&lt;/sub&gt;)</td>
<td>0.34 kg</td>
</tr>
<tr>
<td></td>
<td>Offset from motor axes (R)</td>
<td>0.06 m</td>
</tr>
<tr>
<td></td>
<td>Flexural rigidity (EI)</td>
<td>72.2 Nm²</td>
</tr>
<tr>
<td>Payloads</td>
<td>Mass (M&lt;sub&gt;p&lt;/sub&gt;)</td>
<td>variable from 0.075 kg to 0.753 kg</td>
</tr>
<tr>
<td>Hub</td>
<td>Horizontal inertia (J&lt;sub&gt;hp&lt;/sub&gt;)</td>
<td>0.468 kgm²</td>
</tr>
<tr>
<td></td>
<td>Vertical inertia (J&lt;sub&gt;vh&lt;/sub&gt;)</td>
<td>0.249 kgm²</td>
</tr>
<tr>
<td>Torque motors</td>
<td>Maximum torque</td>
<td>33 Nm</td>
</tr>
<tr>
<td></td>
<td>Rotor inertia</td>
<td>0.013 kgm²</td>
</tr>
<tr>
<td></td>
<td>Torque constant</td>
<td>1.2 Nm/A</td>
</tr>
<tr>
<td></td>
<td>Maximum control signal</td>
<td>±8.5 V</td>
</tr>
<tr>
<td>Encoders</td>
<td>Line count</td>
<td>3600</td>
</tr>
<tr>
<td></td>
<td>Resolution (x4 counting)</td>
<td>4.4×10⁻⁴ rad</td>
</tr>
<tr>
<td>Potentiometers</td>
<td>Linearity</td>
<td>±0.2%</td>
</tr>
<tr>
<td></td>
<td>Linearity wrt link tip position</td>
<td>±3×10⁻³ rad</td>
</tr>
<tr>
<td>Interface</td>
<td>A/D (0-10V differential inputs)</td>
<td>14 bit</td>
</tr>
<tr>
<td></td>
<td>D/A (±10V outputs)</td>
<td>16 bit</td>
</tr>
<tr>
<td></td>
<td>Encoder interface card</td>
<td>24 bit</td>
</tr>
</tbody>
</table>

The flexible link is an homogeneous, cylindrical, aluminium rod with constant properties along its length. The motion of the link is not artificially constrained in any plane. The link is symmetrical about its longitudinal axis, and this axis intersects both horizontal and vertical drive axes at the hub. Each of the interchangeable payloads is symmetrical about the longitudinal axis of the link, with centre of gravity positioned coincident with the link tip.

The vertical and horizontal displacements of the tip of the link are measured. Each measurement is composed of two parts as shown in Figure 3. Firstly, the position of the link tip in relation to the hub axis is measured using a precision potentiometer (θ<sub>hub</sub>). This is combined with the angle of the hub relative to earth axes, measured by an incremental optical encoder (θ<sub>hub</sub>). These signals are interfaced to a PC which is used to implement control algorithms.

MODELLING

Link element model

The Newton-Euler method for modelling transverse vibrations in beams will be used. Only motion in the vertical plane will be considered. The approach broadly follows that of Meirovitch [8], with extensions to accommodate gravitational effects. A three dimensional free body diagram of a small element, length Δx, of the flexible link is shown in Figure 4.

In the figure, shear forces are denoted by V<sub>z</sub>, and bending moments by M<sub>z</sub>. Neglecting angular inertia and shear deformation of the element, and neglecting second order effects, the following link element equation for vertical motion can be derived:

\[-mg_n - EI \frac{\partial^4 Z(x,t)}{\partial x^4} = m \frac{\partial^2 Z(x,t)}{\partial t^2}\] (1)

where Z(x,t) is the vertical displacement of the element in earth axes, and g<sub>n</sub> is the gravitational component normal to the axis of the element. The angular motion of the link can be represented by linear motion of its elements if the angular deviation is small.
Boundary conditions

Four boundary conditions can be derived. There is a geometric boundary condition at the root of the link:

\[ Z(x, t)|_{x=0} = R \frac{\partial Z(x, t)}{\partial x} \bigg|_{x=0} \]  

(2)

Taking moments about the hub axis gives a second boundary condition at the root of the link:

\[ T_{m}(t) - REl \frac{\partial^3 Z(x, t)}{\partial x^3} \bigg|_{x=0} = EI \frac{\partial^2 Z(x, t)}{\partial x^2} \bigg|_{x=0} - T_{g} \]  

(3)

where \( T_{m} \) is the motor torque, \( T_{g} \) is the torque required to support the manipulator weight, and \( J_{lc} \) is the hub inertia in the vertical plane.

Two boundary conditions are associated with the payload. The centre of gravity of the payload lies on the link axis at \( x=L \). Resolving forces acting on the payload vertically:

\[ EI \frac{\partial^3 Z(x, t)}{\partial x^3} \bigg|_{x=L} = M_{p} \frac{\partial^2 Z(x, t)}{\partial x^2} \bigg|_{x=L} - M_{p} \cdot g \sin \alpha \]  

(4)

where \( M_{p} \) is the mass of the payload. Finally, considering the moments acting about the centre of gravity of the payload:

\[ -EI \frac{\partial^2 Z(x, t)}{\partial x^2} \bigg|_{x=L} = J_{p} \frac{\partial^3 Z(x, t)}{\partial x^3} \bigg|_{x=L} \]  

(5)

where \( J_{p} \) is the moment of inertia of the payload.

Mode shapes

A set of mode shapes will be determined by considering the free response of the system. These will then be used to develop the solution to the forced vibration problem.

The free response is defined by element equation (1) and boundary conditions (2) to (5), with the motor and gravity terms set to zero. A solution to these equations is assumed to have the form:

\[ Z(x, t) = S_{z}(x)Q_{z}(t) \]  

(6)

Substituting this solution into equation (1), and neglecting the gravity forcing term, allows the variables to be separated:

\[ \frac{1}{Q_{z}(t)} \frac{d^2 Q_{z}(t)}{dt^2} - \frac{EI}{m} \frac{d^4 S_{z}(x)}{dx^4} = 0 \]  

(7)

For the two sides of the equation (7) to be equal for all values of \( x \) and \( t \), they must be constant. Let this constant be \(-\omega_{z}^{2}\), where \( \omega_{z} \) is real. Hence:

\[ \frac{d^2 Q_{z}(t)}{dt^2} + \omega_{z}^2 Q_{z}(t) = 0 \]  

(8)

\[ \frac{d^4 S_{z}(x)}{dx^4} - \frac{\omega_{z}^2 m}{EI} S_{z}(x) = 0 \]  

(9)

Equation (8) represents a second order system with zero damping and natural frequency \( \omega_{z} \). Equation (9) describes the amplitude variation along the link, i.e. the mode shape. The boundary conditions for this differential equation are derived by substituting equation (6) into (2) to (5):

\[ S_{z}(x)|_{x=0} = R \frac{dS_{z}(x)}{dx} \bigg|_{x=0} \]  

(10)

\[ -REl \frac{d^3 S_{z}(x)}{dx^3} \bigg|_{x=0} + EI \frac{d^2 S_{z}(x)}{dx^2} \bigg|_{x=0} = -J_{lc} \omega_{z}^2 \frac{dS_{z}(x)}{dx} \bigg|_{x=0} \]  

(11)

\[ EI \frac{d^3 S_{z}(x)}{dx^3} \bigg|_{x=L} = -M_{p} \omega_{z}^2 S_{z}(x) \bigg|_{x=L} \]  

(12)

\[ EI \frac{d^2 S_{z}(x)}{dx^2} \bigg|_{x=L} = J_{p} \omega_{z}^2 \frac{dS_{z}(x)}{dx} \bigg|_{x=L} \]  

(13)

Solving equation (9) with these boundary conditions gives a set of mode shapes. Note that mode 0 can be interpreted as a rigid mode, i.e. the solution of these equations obtained with \( \omega_{z0} = 0 \).

State space model

The general solution of equation (1) with boundary conditions (2) to (5) is a sum of modal terms:

\[ Z(x, t) = \sum_{i=0}^{\infty} S_{z}(x)Q_{z}(t) \]  

(14)

Substituting equation (14) into (1):

\[ \sum_{i=0}^{\infty} \left( mS_{z}(x) \frac{d^2 Q_{z}(t)}{dt^2} + EI \frac{d^4 S_{z}(x)}{dx^4} \right) = -mg_{x} \]  

(15)

\[ \sum_{i=0}^{\infty} \left( \frac{d^2 Q_{z}(t)}{dt^2} + \omega_{z}^2 Q_{z}(t) \right) mS_{z}(x) = -mg_{x} \]  

(16)

\[ \sum_{i=0}^{\infty} \left( \frac{d^2 Q_{z}(t)}{dt^2} + \omega_{z}^2 Q_{z}(t) \right) \int_{0}^{L} mS_{z}(x)S_{y}(x) \, dx \]  

(17)

From equations (9) to (13) it can be shown that the following orthogonality condition holds (9):

\[ m \int_{0}^{L} S_{z}(x)S_{y}(x) \, dx + J_{lc} \frac{dS_{z}(x)}{dx} \bigg|_{x=0} - \frac{dS_{y}(x)}{dx} \bigg|_{x=0} = 0 \]  

(18)

\[ M_{p}S_{z}(x) \bigg|_{x=L} - \frac{dS_{z}(x)}{dx} \bigg|_{x=L} = J_{p} \frac{dS_{y}(x)}{dx} \bigg|_{x=L} \]

(\( \delta_{j} = 0 \) for \( i \neq j \) and \( \delta_{j} = 1 \) for \( i = j \)).
In addition, substituting equation (14) into the boundary conditions (3) to (5) respectively gives:

\[
T_z - T_y = \sum_{i=0}^{\infty} \left( -REI \frac{d^2 S_{2i}(x)}{dx^2} \bigg|_{x=L} + EI \frac{d^2 \tilde{S}_{2i}(x)}{dx^2} \bigg|_{x=L} \right) \tilde{Q}_{2i} 
\]

\[
= \sum_{i=0}^{\infty} J_p \frac{dS_{2i}(x)}{dx} \bigg|_{x=L} \tilde{Q}_{2i} 
\]

(19)

\[
\sum_{i=0}^{\infty} EI \frac{d^3 S_{2i}(x)}{dx^3} \bigg|_{x=L} \tilde{Q}_{2i} - M_p g \sum_{i=0}^{\infty} M_p S_{2i}(x) \bigg|_{x=L} \tilde{Q}_{2i} 
\]

\[
= \sum_{i=0}^{\infty} \bigg[ \sum_{j=0}^{\infty} J_p \frac{dS_{2j}(x)}{dx} \bigg|_{x=L} \bigg] \tilde{Q}_{2j} 
\]

(20)

\[
\sum_{i=0}^{\infty} -EI \frac{d^2 S_{2i}(x)}{dx^2} \bigg|_{x=L} \tilde{Q}_{2i} = \sum_{j=0}^{\infty} J_p \frac{dS_{2j}(x)}{dx} \bigg|_{x=L} \tilde{Q}_{2j} 
\]

(21)

Then substituting (11) into (19) gives:

\[
\sum_{i=0}^{\infty} J_p \frac{dS_{2i}(x)}{dx} \bigg|_{x=L} \frac{dS_{2i}(x)}{dx} \bigg|_{x=L} \left( \tilde{Q}_{2i} + \omega_{2i}^2 \tilde{Q}_{2i} \right) 
\]

\[
= (T_z - T_y) \frac{dS_{2i}(x)}{dx} \bigg|_{x=L} \tilde{Q}_{2i} 
\]

(22)

and substituting (12) into (20) gives:

\[
\sum_{i=0}^{\infty} M_p S_{2i}(x) \bigg|_{x=L} S_{2i}(x) \bigg|_{x=L} \left( \tilde{Q}_{2i} + \omega_{2i}^2 \tilde{Q}_{2i} \right) 
\]

\[
= -M_p g \sum_{i=0}^{\infty} S_{2i}(x) \bigg|_{x=L} \tilde{Q}_{2i} 
\]

(23)

and finally substituting (13) into (21) gives:

\[
\sum_{i=0}^{\infty} J_p \frac{dS_{2i}(x)}{dx} \bigg|_{x=L} \frac{dS_{2i}(x)}{dx} \bigg|_{x=L} \left( \tilde{Q}_{2i} + \omega_{2i}^2 \tilde{Q}_{2i} \right) = 0 
\]

(24)

Substituting the orthogonality condition (18), and equations (22) to (24), into (17) gives:

\[
\frac{d^2 \tilde{Q}_{2i}(t)}{dt^2} + \omega_{2i}^2 \tilde{Q}_{2i}(t) = \left( T_z(t) - T_y \right) S_{2i}(x) \bigg|_{x=L} \frac{R}{R} 
\]

\[
- \sum_{j=0}^{\infty} J_p \frac{dS_{2j}(x)}{dx} \bigg|_{x=L} \int_0^L mg S_{2j}(x) \, dx 
\]

(25)

The influence of the weight of link and payload depends on the relative orientation of the gravity vector. The following approximation will be used:

\[
g_a = g \cos(\theta_{tot}) 
\]

(26)

The angle \( \theta_{tot} \) is the angular displacement of the tip, or payload, above the horizontal. This is a good angle to use as in many configurations of the experimental manipulator it is the payload weight which is by far the most significant.

If the centre of gravity of the hub is at angle \( \theta_g \) above the hub angle, and displaced \( L_g \) from the hub axis, then:

\[
T_g = M_h g L_g \cos(\theta_{hub} + \theta_g) 
\]

(27)

From equation (25) the time responses for the individual modes are independent. It is expected that in general the higher the frequency the less significant the mode’s contribution to the overall response. Hence a finite number of the lower frequency modes, \( i = 0 \) to \( n \), will provide a good approximation. So equation (25) can be used to construct a state space model of finite dimension:

\[
\dot{x} = A x + B u 
\]

(28)

\[
x = [q_{00} \; q_{01} \; q_{10} \; \ldots \; q_{0n} \; q_{1n}]^T 
\]

(29)

where

\[
A = \begin{bmatrix} 
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
-\omega_{20}^2 & -C_{20} & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & -\omega_{2n}^2 & -C_{2n} 
\end{bmatrix} 
\]

(30)

\[
B = \begin{bmatrix} 
\frac{S_{20}(0)}{R} & -M_h g L_g \frac{S_{20}(0)}{R} & -S_{00}(L) M_{Lg} & -\int_{0}^{L} mg S_{20}(x) \, dx \\
0 & 0 & 0 & 0 \\
\frac{S_{21}(0)}{R} & -M_h g L_g \frac{S_{21}(0)}{R} & -S_{10}(L) M_{Lg} & -\int_{0}^{L} mg S_{21}(x) \, dx \\
0 & 0 & 0 & 0 \\
\frac{S_{2n}(0)}{R} & -M_h g L_g \frac{S_{2n}(0)}{R} & -S_{0n}(L) M_{Lg} & -\int_{0}^{L} mg S_{2n}(x) \, dx 
\end{bmatrix} 
\]

(31)

\[
u = \begin{bmatrix} 
T_z(t) \\
\cos(\theta_{hub} + \theta_g) \\
\cos(\theta_{tot}) 
\end{bmatrix} 
\]

(32)

and where \( C_{2i} \) are damping coefficients included to represent the low levels of structural and mechanical damping which will be present.

The following output equation defines three system outputs: the total angular deflection of the link, the hub angle, and the angular deflection due to the flexible motion of the link alone, respectively,

\[
y = C_x x 
\]

(33)

\[
y = [\theta_{tot} \; \theta_{hub} \; \theta_{flex}]^T 
\]

(34)
MODEL VERIFICATION
Simulated and actual responses can be compared to verify this model. As the open-loop plant is only marginally stable, a comparison of open-loop responses is impractical. Hence closed-loop control is required, and a PIV (proportional plus integral plus velocity feedback) controller is used for this purpose. A comparison between simulated and experimental link tip position responses in the vertical plane is shown in Figure 5. Note that three modes \( n = 0, 1, 2 \) are simulated, and the payload is 0.414 kg.

A good correlation between the simulated and experimental responses is seen. It is found that including more modes in the model gives no improvement in the simulated response [9], and so it can be deduced that a three mode model will be adequate for controller design.

\[
C_z = \begin{bmatrix}
    \frac{S_0(L)}{(L+R)} & 0 & \frac{S_0(L)}{(L+R)} & 0 \\
    \frac{S_0(0)}{R} & 0 & \frac{S_0(0)}{R} & 0 \\
    \frac{S_0(L)}{(L+R)} - \frac{S_0(0)}{R} & 0 & \frac{S_0(L)}{(L+R)} - \frac{S_0(0)}{R} & 0
\end{bmatrix}
\]

\( (35) \)

\[ \]

H∞ CONTROLLER DESIGN
The PIV controller has been designed, by trial and error, to give the best achievable compromise between speed of response, settling time, and steady-state error. However it is apparent that the response is far from acceptable. Thus an H∞ controller has also been designed for this manipulator, with the objective of improving the tracking performance whilst in addition guaranteeing stability despite payload variation.

A mixed sensitivity H∞ controller is used. The cost function consists of weighted sensitivity function, weighted complementary sensitivity function, and weighted control sensitivity function. The sensitivity weighting function is chosen to give an integral action controller, using the method of Zhou et al [10]. The controller is designed for a nominal payload. Thus a known modelling error results when other payloads are used. The control sensitivity weighting is chosen to be greater than the maximum additive modelling error at any frequency across the whole frequency range. This enables closed-loop stability to be guaranteed across the whole payload range. The design of the controller is discussed fully by Sutton [9].

Figure 6 presents the step response results with the actual plant being the same as the nominal. The results show that the system has a fast response and a small settling time. The performance of the controller is significantly better than that achieved using the PIV controller. The integral action is seen to give zero steady state error after the initial position offset is eliminated.

The results presented in Figure 7 are the step responses with the whole range of payloads (see Table 1). The payload varies by a factor of ten, so the plant variation is very significant. Although the H∞ controller exhibits stability robustness across this range as designed, there is distinct variation in the achieved level of performance.

CONCLUSIONS
The need for stiff structures in precision dynamic machines has a significant impact on performance. The ability to use lightweight, flexible structures would improve dynamic response, reduce power consumption, and also reduce machine size, weight and cost. Robot manipulators are just one application to which such comments can be applied.
The degree of sophistication required in the controller for a flexible manipulator is dramatically greater than its rigid counterpart. As part of the development towards a control scheme for general flexible manipulators, an experimental manipulator has been constructed, with a single link which can be driven in horizontal and vertical planes. The effect of gravity, almost always excluded from other studies, is included in the modelling of this system. As shown for the vertical plane, a close correlation between the response of the model and that of the experimental system is found. A proportional integral controller with velocity feedback, tuned experimentally, will give a stable closed loop response. However, with this manipulator, the best balance of performance that can be achieved still gives a very oscillatory response. Hence the motivation to design a model-based controller. A mixed sensitivity $H^\infty$ controller not only gives the freedom to manipulate the dynamic response for the nominal plant, but robustness to parameter variation can be explicitly incorporated. In this work the variation of the payload within a known range of masses is considered. The experimental results show the improved performance with the $H^\infty$ controller, and verify that stability is maintained with payload mass variation.

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**References**